

Singular conformally invariant trilinear forms and covariant differential operators on the sphere

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Abstract

Let $G = SO_0(1, n)$ be the conformal group acting on the $(n - 1)$ dimensional sphere S , and let $(\pi_\lambda)_{\lambda \in \mathbb{C}}$ be the spherical principal series. For generic values of $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ in \mathbb{C}^3 , there exists a (essentially unique) trilinear form on $\mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S)$ which is invariant under $\pi_{\lambda_1} \otimes \pi_{\lambda_2} \otimes \pi_{\lambda_3}$. Using differential operators on the sphere S which are covariant under the conformal group $SO_0(1, n)$, we construct new invariant trilinear forms corresponding to singular values of λ . The new forms are shown to be certain residues of the family of generic forms (which depends meromorphically on λ).

Soit $G = SO_0(1, n)$ le groupe conforme de la sphère S de dimension $(n - 1)$. Soit $(\pi_\lambda)_{\lambda \in \mathbb{C}}$ la série principale sphérique associée. Pour des valeurs génériques de $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ dans \mathbb{C}^3 , il existe une forme trilinéaire (essentiellement unique) sur $\mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S)$ qui est invariante par $\pi_{\lambda_1} \otimes \pi_{\lambda_2} \otimes \pi_{\lambda_3}$. En utilisant les opérateurs différentiels sur S qui sont conformément covariants, nous construisons de nouvelles formes trilinéaires invariantes correspondant des valeurs singulières de λ . On montre que ces nouvelles formes sont des résidus de la famille des formes génériques (qui dépend méromorphiquement de λ).

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Introduction

Let $G = SO_0(1, n)$, acting conformally on the $(n - 1)$ -dimensional sphere S . In a previous paper, coauthored with B. Ørsted (see [1]), we introduced a trilinear form on $\mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S)$ which is invariant under $\pi_{\lambda_1} \otimes \pi_{\lambda_2} \otimes \pi_{\lambda_3}$, where $\lambda_1, \lambda_2, \lambda_3$ are three complex parameters, and for $\lambda \in \mathbb{C}$, π_λ is the spherical (nonunitary) principal series representation of $SO_0(1, n)$,

realized on $\mathcal{C}^\infty(S)$. The form is constructed via meromorphic continuation in the parameter $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ in \mathbb{C}^3 , with explicitly described simple poles. In the present paper, we determine (some of) the residues of this meromorphic family, producing new conformally invariant trilinear forms. Their expressions involve *covariant differential operators* on $\mathcal{C}^\infty(S)$.

The geometric and functional context is presented in section 1. The two next sections (sections 2, 3) are devoted to a brief study of covariant differential operators on the sphere, for which we could not find an adequate reference. In section 2, we give (following Kostant and Lepowsky) necessary conditions for the existence of such a covariant operator. In section 3, we study the meromorphic continuation (in the parameter s) of the distribution $|x - y|^s$ on $S \times S$. It has simple poles at $s = -(n - 1) - 2k, k \in \mathbb{Z}$ and its residues at the poles are determined, leading to the construction of (all) covariant differential operators on S . In section 4, we use these covariant differential operators to construct new families of trilinear invariant forms on $\mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S)$. Viewed as distributions on $S \times S \times S$, they are *singular* (supported in lower dimensional submanifolds). The families are indexed by an integer k in \mathbb{N} , and depend meromorphically on two complex parameters. In section 5 we first recall the construction (cf [1]) of the meromorphic family of trilinear forms $(\mathcal{K}_\alpha)_{\alpha \in \mathbb{C}^3}$. The set of poles is a union of planes in \mathbb{C}^3 , coming in four families of parallel and equidistant planes. The new forms are shown to be (essentially) the residues along the planes belonging to three of the four families. The residues corresponding to the last family are even more singular (the corresponding distributions are supported by the "diagonal" in $S \times S \times S$) and we delete their study to a (hopefully) near future.

1 The geometric context

Let $S = S^{n-1}$ be the unit sphere in \mathbb{R}^n ,

$$S = \{x = (x_1, x_2, \dots, x_n) ; |x| = \langle x, x \rangle^{\frac{1}{2}} = (x_1^2 + x_2^2 + \dots + x_n^2)^{\frac{1}{2}} = 1\}$$

equipped with the Riemannian metric naturally induced from the Euclidean structure of \mathbb{R}^n . We assume $n \geq 3$, although most of the statements are (or could be made) valid for $n = 2$ (see [1] for similar remarks).

The group $K = SO(n, \mathbb{R})$ operates transitively on S by isometries. However, our interest is in conformal geometry of the sphere, for which another model of the sphere is more fitted. Let $\mathbb{R}^{1,n}$ be the real vector space of

dimension $n + 1$ equipped with the Lorentzian quadratic form

$$[y, y] = y_0^2 - (y_1^2 + \cdots + y_n^2)$$

and let \mathcal{S} be the set of all isotropic lines in $\mathbb{R}^{1,n}$, viewed as a closed submanifold of the real n -dimensional projective space. Then the mapping

$$x \mapsto \mathbb{R}(1, x) \quad S \longmapsto \mathcal{S}$$

is a 1-1 correspondance, which is easily seen to be a diffeomorphism. The group $G = SO_o(1, n)$ operates naturally on \mathcal{S} , and this action can be transferred to an action of G on S . If x is an element of S , and g belongs to G , then $g(x)$ is determined by the following equality in $\mathbb{R}^{1,n}$:

$$(1, g(x)) = \frac{1}{(g.(1, x))_0} g.(1, x) .$$

The group K can be viewed as a closed subgroup of G and is a maximal compact subgroup of G . Introduce the base point $\mathbf{1} = (1, 0, \dots, 0)$ in S . Then the stabilizer of $\mathbf{1}$ in G is the parabolic subgroup $P = MAN$, where M is the stabilizer of $\mathbf{1}$ in K (isomorphic to $SO(n - 1)$), A is the Abelian 1-dimensional subgroup given by

$$A = \left\{ a_t = \begin{pmatrix} \cosh t & \sinh t & 0 & \dots & 0 \\ \sinh t & \cosh t & 0 & \dots & 0 \\ 0 & 0 & 1 & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & 1 \end{pmatrix}, \quad t \in \mathbb{R} \right\}$$

and

$$N = \left\{ n_\xi = \begin{pmatrix} 1 + \frac{|\xi|^2}{2} & -\frac{|\xi|^2}{2} & \xi^t & & \\ \frac{|\xi|^2}{2} & 1 - \frac{|\xi|^2}{2} & \xi^t & & \\ & & 1 & & \\ \xi & -\xi & & \ddots & \\ & & & & 1 \end{pmatrix}, \quad \xi \in \mathbb{R}^{n-1} \right\} .$$

The element a_t ($t \in \mathbb{R}$) acts on S by

$$a_t \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{\sinh t + x_1 \cosh t}{\cosh t + x_1 \sinh t} \\ \frac{x_2}{\cosh t + x_1 \sinh t} \\ \vdots \\ \frac{x_n}{\cosh t + x_1 \sinh t} \end{pmatrix} .$$

Let $\mathfrak{g} = \text{Lie}(G)$ be the Lie algebra of G , and let $\mathfrak{k}, \mathfrak{m}, \mathfrak{a}, \mathfrak{n}$ be the Lie subalgebras of (respectively) K, M, A, N . Let

$$H = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \in \mathfrak{gl}(n+1, \mathbb{R})$$

be a generator of \mathfrak{a} .

Let $\bar{\mathfrak{n}} = \theta(\mathfrak{n})$, where θ is the standard Cartan involution of \mathfrak{g} given by $\theta X = -X^t$. The Lie algebra \mathfrak{g} admits the following decomposition

$$\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} ,$$

which is the eigenspace decomposition for the action of $\text{ad } H$, with eigenvalue 0 on $\mathfrak{m} \oplus \mathfrak{a}$, 1 on \mathfrak{n} and -1 on $\bar{\mathfrak{n}}$. Let also $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the parabolic subalgebra corresponding to P .

The action of G turns out to be *conformal*, that is, for any g in G and x in S , the differential $Dg(x)$ satisfies, for any ξ in the tangent space $T_x S$

$$|Dg(x)\xi| = \kappa(g, x)|\xi| , \quad (1)$$

where $\kappa(g, x)$ is a positive constant, called the *conformal factor* of g at x .

Proposition 1.1. *The conformal factor $\kappa(g, x)$ is a smooth function of both g and x , which satisfies moreover the following properties :*

i) (cocycle property)

$$\forall g_1, g_2 \in G, x \in S, \quad \kappa(g_1 g_2, x) = \kappa(g_1, g_2(x)) \kappa(g_2, x) \quad (2)$$

$$ii) \quad \forall g \in G, x \in S \quad \kappa(g, g^{-1}(x)) = \kappa(g^{-1}, x)^{-1}$$

$$iii) \quad \forall x \in S, k \in K \quad \kappa(k, x) = 1$$

$$iv) \quad \forall x \in S, t \in \mathbb{R}, \quad \kappa(a_t, x) = (\cosh t + x_1 \sinh t)^{-1}.$$

Let g in G . As the dimension of the tangent space $T_x S$ is $n - 1$, the Jacobian of g at x (with respect to the Euclidian measure on S) is given by

$$j(g, x) = \kappa(g, x)^{n-1} .$$

The corresponding change of variable formula is

$$\int_S f(g^{-1}(x)) d\sigma(x) = \int_S f(y) \kappa(g, y)^{n-1} d\sigma(y) , \quad (3)$$

where $d\sigma$ is the Lebesgue measure on S .

The Euclidean distance, restricted to $S \times S$, satisfies an important covariance property under the action of G .

Proposition 1.2. *Let g in G and x, y in S . Then*

$$|g(x) - g(y)| = \kappa(g, x)^{\frac{1}{2}} |x - y| \kappa(g, y)^{\frac{1}{2}} . \quad (4)$$

Notice that (1) can be viewed as the infinitesimal form of (4).

2 Conformal representations, associated Verma modules and covariant differential operators

To the conformal action of G on S is associated a family of representations. For convenience set $\rho = \frac{n-1}{2}$. Now let λ be any complex number and define for any g in G and f in $\mathcal{C}^\infty(S)$

$$\pi_\lambda(g)f(x) = \kappa(g^{-1}, x)^{\rho+\lambda} f(g^{-1}(x)) \quad (5)$$

for x in S . Thanks to the cocycle property of κ (see (2)), π_λ is a representation of G on the space $\mathcal{C}^\infty(S)$, which is continuous for the usual topology on $\mathcal{C}^\infty(S)$. For a detailed study of these representations, see [9]. They are also called the *spherical (non unitary) principal series* of G .

The reason for putting ρ in the parameter of the representation is to introduce more symmetry in the formulation of the following duality result.

Proposition 2.1. *For f and φ in $\mathcal{C}^\infty(S)$, and for any $g \in G$,*

$$\int_S \pi_\lambda(g)f(x)\varphi(x)d\sigma(x) = \int_S f(x)\pi_{-\lambda}(g^{-1})\varphi(x)d\sigma(x) . \quad (6)$$

This is a consequence of the change of variable formula (3).

This can be used to extend the definition of the representation π_λ to the space of distributions $\mathcal{C}^{-\infty}(S)$ by setting for F in $\mathcal{C}^{-\infty}(S)$

$$(\pi_\lambda(g)F, \varphi) := (F, \pi_{-\lambda}(g^{-1})\varphi) \quad (7)$$

for any φ in $\mathcal{C}^\infty(S)$. Let $\delta = \delta_{\mathbf{1}}$ be the Dirac measure at $\mathbf{1}$. Then

$$\pi_\lambda(g)\delta = \kappa(g, \mathbf{1})^{\rho-\lambda} \delta_{g(\mathbf{1})} . \quad (8)$$

As usual, it is possible to differentiate the action of G on $\mathcal{C}^\infty(S)$ and obtain a representation (still denoted by π_λ) of the Lie algebra \mathfrak{g} or of the

universal envelopping algebra $\mathfrak{U}(\mathfrak{g})$. The envelopping algebra acts by differential operators on $\mathcal{C}^\infty(S)$, and also on $\mathcal{C}^{-\infty}(S)$. Hence this action preserves the support of the distributions.

Let \mathcal{V} be the space of distributions on S supported at $\{\mathbf{1}\}$. The enveloping algebra $\mathfrak{U}(\mathfrak{g})$ acts on \mathcal{V} through π_λ , and we let \mathcal{V}_λ be the corresponding $\mathfrak{U}(\mathfrak{g})$ -module. By differentiation of (8),

$$\pi_\lambda(X)\delta = 0, \quad \text{for } X \in \mathfrak{m} \oplus \mathfrak{n} \quad (9)$$

$$\pi_\lambda(H)\delta = (\lambda - \rho)\delta \quad (10)$$

Moreover, by the Poincaré-Birkhoff-Witt theorem, the space \mathcal{V}_λ is identified with $\mathfrak{U}(\bar{\mathfrak{n}})\delta$. This remark, together with (9) and (10) show that \mathcal{V}_λ is a *generalized Verma module*, in the sense of Lepowsky (see [7], [8], or [3]).

Let λ, μ be two complex numbers. A differential operator D (with smooth coefficients) on S is said to be *conformally covariant* with respect to the representations (π_λ, π_μ) if, for any g in G

$$D \circ \pi_\lambda(g) = \pi_\mu(g) \circ D. \quad (11)$$

Proposition 2.2. *Let D be a differential operator on S , which is not identically 0, and assume D is conformally covariant w.r.t. (π_λ, π_μ) . Then, either $\lambda = \mu$ (and then D is a multiple of the identity), or $\mu = -\lambda = k$ for some k in \mathbb{N} .*

Proof. Let D be such differential operator. As D commutes to the action of K (which does not depend on λ), D is a polynomial in the Laplacian Δ (a special case of the characterization of the invariant differential operators on a Riemannian symmetric space). As $D \neq 0$, $D\delta$ is already different from 0. Now D acts on distributions on S , preserving the support. Hence D yields by restriction an operator on \mathcal{V} and the covariance property (11) shows that D induces a homomorphism of $\mathfrak{U}(\mathfrak{g})$ -modules from \mathcal{V}_λ into \mathcal{V}_μ . Moreover, $D\delta$ is a non trivial conical vector (i.e. killed by $\mathfrak{m} \oplus \mathfrak{n}$) of weight $\lambda - \rho$ in \mathcal{V}_μ . Now the homomorphisms of generalized Verma modules and the conical vectors were studied by Lepowsky (see [7], [8]) and the only possibilities for the existence of a non trivial homomorphism between \mathcal{V}_λ and \mathcal{V}_μ are $\lambda = \mu$ or $\mu = -\lambda = k$, for some k in \mathbb{N} ([8]). \square

Remark. The quoted reference shows that the necessary conditions on λ and μ for the existence of a homomorphism from \mathcal{V}_λ into \mathcal{V}_μ are also sufficient. Moreover the homomorphism is then unique, up to a constant. For any k in \mathbb{N} , we will now construct an explicit covariant differential operator on S which yields a non trivial homomorphism of \mathcal{V}_{-k} into \mathcal{V}_k .

3 The meromorphic continuation of $|1 - x|^s$ and its residues

For s in \mathbb{C} , let $h_s(x) = |1 - x|^s$. This defines a smooth function on S outside of the point $\mathbf{1}$. For $\Re s > -(n - 1)$, the function is integrable and will be considered as a distribution on S , still denoted by h_s . It depends holomorphically on s . We want to show that it can be extended meromorphically to \mathbb{C} . The main ingredient to do this is the *Bernstein-Sato identity* (*stricto sensu*, one should consider the Bernstein-Sato identity for the smooth function $|1 - x|^2$ on S).

Proposition 3.1. *The following identity holds on $S \setminus \{\mathbf{1}\}$*

$$[\Delta + \frac{s}{2}(\frac{s}{2} + n - 2)] h_s = s(s + n - 3) h_{s-2} \quad (12)$$

Proof. A function f on S which is invariant under the subgroup M depends only on the distance from x to $\mathbf{1}$ and can be written in a unique way as $f(x) = \varphi(\theta)$, where $\theta = \arccos \langle x, \mathbf{1} \rangle$, and φ is a function defined on the interval $[0, \pi]$. As the Laplacian commutes to the rotations, the function Δf is also invariant under M , and the following relation holds :

$$\Delta f(x) = \varphi''(\theta) + \frac{(n-2)}{\tan \theta} \varphi'(\theta) .$$

As $|1 - x|^2 = 2(1 - \cos \theta)$, (12) follows easily. \square

Proposition 3.2. *The function $s \mapsto h_s$ originally defined for $\Re s > -(n - 1)$ can be extended as a (distribution-valued) meromorphic function on \mathbb{C} , with simple poles at $s = -(n - 1) - 2k, k \in \mathbb{N}$.*

The proof is standard and uses mainly integration by parts in the form

$$\left(h_s, \left(\Delta + \frac{s}{2}(\frac{s}{2} + n - 2) \right) f \right) = \left(\left(\Delta + \frac{s}{2}(\frac{s}{2} + n - 2) \right) h_s, f \right) ,$$

where f is any smooth test function. If the left handside is already defined, and $s(s + n - 3) \neq 0$, it can be used to define (h_{s-2}, f) . The meromorphic dependance on s is easy to verify.

A variant of Proposition 3.2 will be used later on.

Proposition 3.3. *Let φ be in $\mathcal{C}^k(S)$ (i.e. k -times continuously differentiable), then the function $s \mapsto \int \varphi(x) h_s(x) d\sigma(x)$ can be extended meromorphically in the open set $\Re(s) < -(n - 1) - 2[\frac{k}{2}]$ with at most simple poles at $s = -(n - 1) - 2l, l \in \mathbb{N}, l < [\frac{k}{2}]$.*

Denote by r_k the residue at $-(n-1) - 2k$ of the distribution-valued function $s \mapsto h_s$.

Proposition 3.4.

$$r_0 := \text{Res}(h_s, -(n-1)) = \frac{\pi^\rho}{\Gamma(\rho)} \delta_1 . \quad (13)$$

Proof.

$$\begin{aligned} (h_s, f) &= \int_S f(x) |\mathbf{1} - x|^s d\sigma(x) \\ &= \int_S (f(x) - f(\mathbf{1})) |\mathbf{1} - x|^s d\sigma(x) + f(\mathbf{1}) \int_S |\mathbf{1} - x|^s d\sigma(x) . \end{aligned}$$

Now $|f(x) - f(\mathbf{1})| \leq C|x - \mathbf{1}|$, so that the first integral is absolutely convergent for s in a (small) neighborhood of $-(n-1)$. Hence, the contribution to the residue at $-(n-1)$ of the first integral is 0. Now

$$\int_S |\mathbf{1} - x|^s d\sigma(x) = 2^{n-1} \pi^\rho 2^s \frac{\Gamma(\frac{s}{2} + \rho)}{\Gamma(\frac{s}{2} + 2\rho)} .$$

The function on the right handside is meromorphic, has a simple pole at $s = -(n-1) = -2\rho$, with residue equal to $\frac{\pi^\rho}{\Gamma(\rho)}$. \square

Proposition 3.5.

$$r_1 := \text{Res}(h_s, -(n-1) - 2) = \frac{\pi^\rho}{4\Gamma(\rho+1)} \Delta_1 \delta_1 , \quad (14)$$

where

$$\Delta_1 = \Delta - \frac{1}{4}(n-1)(n-3)$$

is the conformal Laplacian or Yamabe operator on S .

Proof. The Bernstein-Sato identity (12) can be extended meromorphically to \mathbb{C} . Taking residue of both sides at $-(n-1)$ yields

$$2(n-1)\text{Res}(h_s, -(n-1) - 2) = [\Delta - (\frac{n-1}{2})(\frac{n-3}{2})] \text{Res}(h_s, -(n-1)) ,$$

which, via (13) gives (14). \square

For any k in \mathbb{N} , introduce the differential operator Δ_k on S given by

$$\Delta_k = \prod_{j=1}^k (\Delta - (\rho + j - 1)(\rho - j)) = \prod_{j=1}^k (\Delta_1 + j(j - 1)) . \quad (15)$$

Observe that Δ_k is a polynomial of degree k in Δ which is of the form $\Delta^k +$ lower order terms. Observe that Δ_k is essentially selfadjoint.

Proposition 3.6. *For k any positive integer,*

$$r_k = \frac{\pi^\rho}{4^k \Gamma(\rho + k) \Gamma(k + 1)} \Delta_k \delta_1 .$$

Proof. Compute the residue at $s = -(n - 1) - 2k$ of both sides of the Bernstein identity (12) to get

$$r_{k+1} = \frac{1}{4(\rho + k)(k + 1)} (\Delta - (\rho + k)(\rho - k - 1)) r_k .$$

Hence the result. \square

Normalize the Haar measure dk on K such that, for any integrable function f on S

$$\int_K f(k \mathbf{1}) dk = \int_S f(x) d\sigma(x) .$$

Let f be a smooth function on S . Let f^\sharp (resp. g^\sharp) be the function on K defined by $f^\sharp(k) = f(k \mathbf{1})$. For f and g two smooth functions on S , define the convolution $f^\sharp \star g^\sharp$ is defined (as a function on K) by

$$(f^\sharp \star g^\sharp)(k) = \int_K f^\sharp(l^{-1}k) g^\sharp(l) dl .$$

It is a smooth function on K , which is right invariant by M , hence defines a function on S , denoted by $f \star g$. Suppose moreover that f is invariant by M , hence of the form $f(x) = F(\langle x, \mathbf{1} \rangle)$. Then, $f \star g$ is given by

$$(f \star g)(x) = \int_S F(\langle x, y \rangle) g(y) d\sigma(y) . \quad (16)$$

The convolution of two functions in $\mathcal{C}^\infty(S)$ is in $\mathcal{C}^\infty(S)$ and the convolution can be extended to distributions on S .

For α a complex parameter, introduce the kernel k_α on $S \times S$ by

$$k_\alpha(x, y) = |x - y|^{-\rho + \alpha} ,$$

and the corresponding operator (formally defined by)

$$K_\alpha f(x) = \int_S k_\alpha(x, y) f(y) d\sigma(y) .$$

To make proper sense, the operator K_α can be reinterpreted as a convolution. As, for x, y in S , $|x - y| = 2(1 - \langle x, y \rangle)$, (16) implies

$$K_\alpha f = h_{-\rho+\alpha} \star f .$$

Using the standard continuity properties of the convolution on S and the results of section 2, it is easy to analytically continue the map $\alpha \mapsto K_\alpha$ on \mathbb{C} , with simple poles at $\alpha = -\rho - 2k, k \in \mathbb{N}$. The corresponding residues are easily computed. In fact, let $R_k = \text{Res}(K_\alpha, -\rho - 2k)$. Then

$$R_k f = r_k \star f = \frac{\pi^\rho}{4^k \Gamma(\rho + k) \Gamma(k + 1)} \Delta_k f . \quad (17)$$

Now, for φ, ψ two functions in $\mathcal{C}^\infty(S)$,

$$(k_\alpha, \varphi \otimes \psi) = \int_{S \times S} k_\alpha(x, y) \varphi(x) \psi(y) d\sigma(x) d\sigma(y) = (K_\alpha \varphi, \psi) .$$

As above, this bilinear form on $\mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S)$ can be meromorphically continued on \mathbb{C} , with same poles as above. But by Schwartz nuclear theorem, this is equivalent to say that the $\alpha \mapsto k_\alpha$ can be meromorphically continued as a distribution (on $S \times S$)-valued function, with same poles as above, and the residues are also easy to calculate.

Proposition 3.7. *Let f be in $\mathcal{C}^\infty(S \times S)$. Then the expression*

$$\iint_{S \times S} f(x, y) |x - y|^{-\rho+\alpha} d\sigma(x) d\sigma(y) \quad (18)$$

originally defined for $\Re \alpha$ large enough can be continued meromorphically to \mathbb{C} , with simple poles at $\alpha = -\rho - 2k, k \in \mathbb{N}$. The residue at $\alpha = -\rho - 2k$ is given by

$$\int_S R_k^{(1)} f(x, x) d\sigma(x) ,$$

where $R_k^{(1)}$ stands for the differential operator R_k acting on the first variable.

Remark 1. We will need a slightly stronger version of this result. Let k be in \mathbb{N} . Then it is possible to choose ℓ large enough so that, if f is merely

in $\mathcal{C}^\ell(S \times S)$, then the integral (18) can be meromorphically continued to $\Re(\alpha) > -\rho - 2k$. This result is proved exactly the same way, but using Proposition 3.3 instead of Proposition 3.2.

Remark 2. The operator R_k is symmetric, so that the residue can also be written as $\int_S R_k^{(2)} f(x, x) d\sigma(x)$, that is by letting R_k act on the second variable.

An important property of the operators K_α is that it is a family of intertwining operators for the representations constructed earlier (the Knapp-Stein intertwining operators for the spherical principal series). In fact, by a change of variable (and meromorphic continuation), one shows that for any g in G

$$K_{-\rho+2\lambda} \circ \pi_\lambda(g) = \pi_{-\lambda}(g) \circ K_{-\rho+2\lambda} .$$

Hence, taking residue on both sides at $\lambda = -k$

$$R_k \circ \pi_{-k}(g) = \pi_k(g) \circ R_k \tag{19}$$

for any g in G . This shows that Δ_k is a covariant differential operator w.r.t. (π_{-k}, π_k) , solving the problem raised at the end of section 2.

For $k = 1$, (19) is a well-known property of the Yamabe operator on the sphere. For higher values of k , it corresponds to the conformal invariance of the Graham-Jenne-Mason-Sparling operators on the sphere (see [4], [5]).

4 Construction of singular conformally invariant trilinear forms

We now begin the heart of this article. In [1], we studied trilinear invariant forms on $\mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S)$. Let $\lambda_1, \lambda_2, \lambda_3$ be three complex numbers and let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. A continuous trilinear form \mathcal{T} on $\mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S)$ is said to be invariant w.r.t. $(\pi_{\lambda_1}, \pi_{\lambda_2}, \pi_{\lambda_3})$ if, for any g in G and f_1, f_2, f_3 in $\mathcal{C}^\infty(S)$

$$\mathcal{T}(\pi_{\lambda_1}(g)f_1, \pi_{\lambda_2}(g)f_2, \pi_{\lambda_3}(g)f_3) = \mathcal{T}(f_1, f_2, f_3) .$$

For generic λ (the exact meaning of "generic" will be given in next section), we constructed a non trivial invariant trilinear form and showed that it is the unique one (up to a constant). Now we deal with the case of singular values of λ .

Let k be in \mathbb{N} , and set for convenience $\alpha_3 = -\rho - 2k$ (notation will be explained in the next section). For α_1, α_2 in \mathbb{C}^2 , define $\mathcal{T}_k = \mathcal{T}_{(\alpha_1, \alpha_2, -\rho-2k)}$

on $\mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S)$ by

$$\begin{aligned} \mathcal{T}_k(f_1, f_2, f_3) = \\ \int_{S \times S} f_3(x_3) f_2(x) \Delta_k[f_1(\cdot)|x_3 - \cdot|^{-\rho+\alpha_2}](x) |x - x_3|^{-\rho+\alpha_1} d\sigma(x) d\sigma(x_3) . \end{aligned} \quad (20)$$

Proposition 4.1. *Let $\alpha = (\alpha_1, \alpha_2, \alpha_3 = -\rho - 2k)$ and let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ satisfying the following relations*

$$\begin{aligned} \alpha_1 &= -\lambda_1 + \lambda_2 + \lambda_3 \\ \alpha_2 &= \lambda_1 - \lambda_2 + \lambda_3 \\ \alpha_3 &= \lambda_1 + \lambda_2 - \lambda_3 . \end{aligned} \quad (21)$$

Then

$$\mathcal{T}_k(\pi_{\lambda_1}(g)f_1, \pi_{\lambda_2}(g)f_2, \pi_{\lambda_3}(g)f_3) = \mathcal{T}_k(f_1, f_2, f_3)$$

for any g in G and $f_1, f_2, f_3 \in \mathcal{C}^\infty(S)$, whenever the integrals make sense.

Proof. We first need a technical lemma. It will be convenient to set, for f_1 in $\mathcal{C}^\infty(S)$ and x_3 in S ,

$$F_{x_3}[f_1](x) = f_1(x) |x_3 - x|^{-\rho+\alpha_2} .$$

Lemma 4.1. *For any g in G , f_1 in $\mathcal{C}^\infty(S)$ and x_3 in S ,*

$$F_{x_3}[\pi_{\lambda_1}(g)f_1] = \kappa(g, y_3)^{-\frac{\rho}{2} + \frac{\alpha_2}{2}} \pi_{-k}(g) F_{y_3}[f_1] , \quad (22)$$

where $x_3 = g(y_3)$.

Proof.

$$\begin{aligned} LHS &= f_1(g^{-1}(x)) |x_3 - x|^{-\rho+\alpha_2} \kappa(g^{-1}, x)^{\rho+\lambda_1} \\ &= f_1(g^{-1}(x)) \kappa(g, y_3)^{-\frac{\rho}{2} + \frac{\alpha_2}{2}} |y_3 - g^{-1}(x)|^{-\rho+\alpha_2} \kappa(g^{-1}, x)^{\frac{\rho}{2} - \frac{\alpha_2}{2} + \rho + \lambda_1} \end{aligned}$$

using (4). Now

$$\frac{\rho}{2} - \frac{\alpha_2}{2} + \rho + \lambda_1 = \rho - k .$$

from which (22) follows. \square

With the notation introduced previously,

$$T_k(f_1, f_2, f_3) = \iint f_3(x_3) f_2(x) \Delta_k F_{x_3}[f_1](x) |x - x_3|^{-\rho+\alpha_1} d\sigma(x) d\sigma(x_3),$$

so that

$$\begin{aligned}
& T_k(\pi_{\lambda_1}(g)f_1, \pi_{\lambda_2}(g)f_2, \pi_{\lambda_3}(g)f_3) = \\
& \iint f_3(g^{-1}(x_3))f_2(g^{-1}(x))\Delta_k\{F_{x_3}[\pi_{\lambda_1}(g)f_1]\}(x)|x-x_3|^{-\rho+\alpha_1}\dots \\
& \dots \kappa(g^{-1}, x)^{\rho+\lambda_2}\kappa(g^{-1}, x_3)^{\rho+\lambda_3}d\sigma(x)d\sigma(x_3)
\end{aligned}$$

Now use (22) and (19), make the change of variables $x = g(y)$ and $x_3 = g(y_3)$ to get

$$\begin{aligned}
& T_k(\pi_{\lambda_1}(g)f_1, \pi_{\lambda_2}(g)f_2, \pi_{\lambda_3}(g)f_3) \\
& = \iint f_3(y_3)f_2(y)\pi_k(g)\Delta_k\{F_{y_3}[f_1]\}(g(y))|y-y_3|^{-\rho+\alpha_1}\dots \\
& \dots \kappa(g, y_3)^{-\frac{\rho}{2}+\frac{\alpha_2}{2}-\frac{\rho}{2}+\frac{\alpha_1}{2}-\rho-\lambda_3+2\rho}\kappa(g, y)^{-\frac{\rho}{2}+\frac{\alpha_1}{2}-\rho-\lambda_2+2\rho}d\sigma(y)d\sigma(y_3) \\
& = \iint f_3(y_3)f_2(y)\Delta_k F_{y_3}[f_1](y)|y-y_3|^{-\rho+\alpha_1}\dots \\
& \kappa(g, y_3)^{-\frac{\rho}{2}+\frac{\alpha_2}{2}-\frac{\rho}{2}+\frac{\alpha_1}{2}-\rho-\lambda_3+2\rho}\kappa(g, y)^{-\frac{\rho}{2}+\frac{\alpha_1}{2}-\rho-\lambda_2+2\rho-\rho-k}d\sigma(y)d\sigma(y_3) .
\end{aligned}$$

Now

$$\begin{aligned}
& -\frac{\rho}{2} + \frac{\alpha_2}{2} - \frac{\rho}{2} + \frac{\alpha_1}{2} - \rho - \lambda_3 + 2\rho = 0 \\
& -\frac{\rho}{2} + \frac{\alpha_1}{2} - \rho - \lambda_2 + 2\rho - \rho - k = 0,
\end{aligned}$$

where in the second line we use the condition $\alpha_3 = -\rho - 2k$. So the last integral reduces to

$$\iint f_3(y_3)f_2(y)\Delta_k F_{y_3}[f_1](y)|y-y_3|^{-\rho+\alpha_1}d\sigma(y)d\sigma(y_3) ,$$

which shows Proposition 4.1. □

Having proved (formally) the invariance of the form \mathcal{T}_k , we now study the convergence of the integral and its meromorphic continuation.

Theorem 4.1. *Let k is in \mathbb{N} . Then the trilinear form $\mathcal{T}_k = \mathcal{T}_{\alpha_1, \alpha_2, -\rho-2k}$ originally defined as a convergent integral for $\Re\alpha_1$ and $\Re\alpha_2$ large enough can be extended meromorphically to \mathbb{C}^2 , with poles along the lines*

$$\alpha_1 + \alpha_2 = 2k - 2l, l \in \mathbb{N} .$$

The trilinear form \mathcal{T}_k is invariant w.r.t. $(\pi_{\lambda_1}, \pi_{\lambda_2}, \pi_{\lambda_3})$, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ is associated to $\boldsymbol{\alpha}$ by the relations (21).

Proof. We first need a technical lemma.

Lemma 4.2. *Let φ be in $\mathcal{C}^\infty(S \times S)$. There exists a (unique) function $\psi(x, y, s)$ which is \mathcal{C}^∞ in x and y and holomorphic in s such that, for all $x \neq y$ in $S \times S$*

$$\Delta_x [|x - y|^s \varphi(x, y)] = |x - y|^{s-2} \psi(x, y, s)$$

Moreover, the function ψ depends continuously on φ .

Proof. Recall the formula

$$\Delta(fg) = \Delta f \cdot g + 2 \overrightarrow{\text{grad}} f \cdot \overrightarrow{\text{grad}} g + f \Delta g .$$

By (12) and the fact that Δ commutes with the action of K

$$\Delta_x(|x - y|^s) = \left(-\frac{s}{2} \left(\frac{s}{2} + n - 2 \right) |x - y|^2 + s(s + n - 3) \right) |x - y|^{s-2} .$$

Moreover,

$$\overrightarrow{\text{grad}}_x(|x - y|^s) = \frac{s}{2} |x - y|^{s-2} \overrightarrow{\text{grad}}_x(|x - y|^2) .$$

The statement of the lemma is a consequence of these three formulæ. \square

Now, by repeated uses of Lemma 4.2, it is possible to write

$$\Delta_k[f_1(\cdot)|x_3 - \cdot|^{-\rho+\alpha_2}](x) = \sum_{\ell=0}^k |x_3 - x|^{-\rho+\alpha_2-2\ell} \psi_\ell(x_3, x, \alpha_2)$$

where the ψ_ℓ are smooth functions on $S \times S$, holomorphic in α_2 , depending continuously on f_1 . Hence $\mathcal{T}_k(f_1, f_2, f_3)$ can be written as a sum of integrals of the form

$$\int_{S \times S} f_2(x) f_3(x_3) \psi_j(x_3, x, s) |x - x_3|^{-2\rho+\alpha_2+\alpha_1-2j} d\sigma(x_3) d\sigma(x)$$

where $j = 0, 1, \dots, k$, and ψ_j is a smooth function on $S \times S$, holomorphic in s and depending continuously on f_1 . Such integrals can be meromorphically continued by use of Proposition 3.7 and the localization of the poles also follows. \square

5 Residues of K_α along a hyperplane of the first kind

Now recall the construction and results of [1]. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be in \mathbb{C}^3 . Consider the kernel K_α on $S \times S \times S$ defined by

$$K_\alpha(x_1, x_2, x_3) = k_{\alpha_1}(x_2, x_3) k_{\alpha_2}(x_3, x_1) k_{\alpha_3}(x_1, x_2)$$

and the associated trilinear form \mathcal{K}_α defined by

$$\mathcal{K}_\alpha(f_1, f_2, f_3) = \int_{S \times S \times S} K_\alpha(x_1, x_2, x_3) f_1(x_1) f_2(x_2) f_3(x_3) d\sigma(x_1) d\sigma(x_2) d\sigma(x_3)$$

Proposition 5.1. *Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be in \mathbb{C}^3 . Then the integral $\mathcal{K}_\alpha(f_1, f_2, f_3)$ is absolutely convergent if $\Re \alpha_j > -\rho, j = 1, 2, 3$ and $\Re(\alpha_1 + \alpha_2 + \alpha_3) > -\rho$. The mapping $\alpha \mapsto \mathcal{K}_\alpha(f_1, f_2, f_3)$ can be extended meromorphically to \mathbb{C}^3 , with simple poles along the planes (in \mathbb{C}^3)*

$$\begin{aligned} \alpha_1 &= -\rho - 2k_1, & k_1 &\in \mathbb{N} \\ \alpha_2 &= -\rho - 2k_2, & k_2 &\in \mathbb{N} \\ \alpha_3 &= -\rho - 2k_3, & k_3 &\in \mathbb{N} \\ \alpha_1 + \alpha_2 + \alpha_3 &= -\rho - 2k, & k &\in \mathbb{N} \end{aligned} \tag{23}$$

For α outside of these planes, the trilinear form $\mathcal{K}_\alpha(f_1, f_2, f_3)$ thus defined is invariant w.r.t. the representations $(\pi_{\lambda_1}, \pi_{\lambda_2}, \pi_{\lambda_3})$, where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ are related by the relations (21).

From the results in [1], more can be said about the residues of \mathcal{K}_α . The poles come into four families of parallel and equidistant planes in \mathbb{C}^3 . The first three (said to be of *type 1*) have some sort of symmetry under a permutation of $\alpha_1, \alpha_2, \alpha_3$, so that it suffices to study the case where $\alpha_3 = -\rho - 2k, k \in \mathbb{N}$. Then the residue is a distribution supported in $\{(x, x, x_3), x, x_3 \in S\}$. For the fourth family (*type 2*), the residue is a distribution supported in the "diagonal" of $S \times S \times S$, i.e. the set $\{x, x, x\}, x \in S$. This last family is not considered in the present paper.

We now proceed to the determination of the residue of the form \mathcal{K}_α at a pole of the form $\alpha = (\alpha_1, \alpha_2, -\rho - 2k)$ for α_1, α_2 generic.

Theorem 5.1. *Let $\alpha^0 = (\alpha_1, \alpha_2, \alpha_3^0)$ be a generic point in the hyperplane $\alpha_3^0 = -\rho - 2k$. Then*

$$\begin{aligned} \text{Res}(K_\alpha(f_1, f_2, f_3), \alpha^0) = \\ \int_{S \times S} f_3(x_3) f_2(x) R_k[f_1(\cdot)|x_3 - \cdot|^{-\rho+\alpha_2}](x) |x - x_3|^{-\rho+\alpha_1} d\sigma(x) d\sigma(x_3) . \end{aligned} \tag{24}$$

More precisely, the right handside integral which converges for $\Re\alpha_1$ and $\Re\alpha_2$ large enough can be extended meromorphically to \mathbb{C}^2 with poles contained in the lines

$$\alpha_1 + \alpha_2 = 2k - 2l, \quad l \in \mathbb{N}.$$

The left handside, a priori defined for α_1, α_2 outside of the lines

$$\alpha_1 = -\rho - 2k_1, \quad \alpha_2 = -\rho - 2k_2, \quad \alpha_1 + \alpha_2 = 2k - 2l_3,$$

$k_1, k_2, l_3 \in \mathbb{N}$, coincides with the left handside.

Proof. As R_k and Δ_k differ by a non vanishing constant, the right handside of (24) depends meromorphically on (α_1, α_2) (see Theorem 4.1) so that, by properties of analytic continuation, it is enough to verify the equality when $\Re\alpha_1$ and $\Re\alpha_2$ are large enough. In this spirit, we have the following technical lemma.

Lemma 5.1. *Let f_3 be in $\mathcal{C}^\infty(S)$. For x_1, x_2 in S , let*

$$F_3(x_1, x_2) = \int_S f_3(x_3) |x_2 - x_3|^{-\rho+\alpha_1} |x_3 - x_1|^{-\rho+\alpha_2} d\sigma(x_3). \quad (25)$$

Let l be in \mathbb{N} . Then for $\Re\alpha_1$ and $\Re\alpha_2$ large enough, the function F_3 is in $\mathcal{C}^l(S \times S)$, and the map $f_3 \mapsto F_3$ is continuous from $\mathcal{C}^\infty(S)$ to $\mathcal{C}^l(S \times S)$.

Proof. Choose, at it is possible, $\Re\alpha_1$ and $\Re\alpha_2$ large enough, so that the kernel

$$|x_2 - x_3|^{-\rho+\alpha_1} |x_3 - x_1|^{-\rho+\alpha_2}$$

is, as a function of three variables, everywhere l -times continuously differentiable. Then the statement follows by the rule of differentiation under integral sign and standard estimates. \square

Now, let f_1, f_2, f_3 be three functions in $\mathcal{C}^\infty(S)$. Let $\Re\alpha_1$ and $\Re\alpha_2$ be large enough, so that the function F_3 defined by (25) is in $\mathcal{C}^l(S \times S)$ for some (large) l . For α_3 near α_3^0 , but $\alpha_3 \neq \alpha_3^0$, the kernel $|x_1 - x_2|^{-\rho+\alpha_3}$ corresponds to a distribution on $S \times S$, depending meromorphically on α_3 and hence (see Lemma 3.7 and the remark inside the proof) the expression

$$\int_{S \times S} f_1(x_1) f_2(x_2) F_3(x_1, x_2) |x_1 - x_2|^{-\rho+\alpha_3} d\sigma(x_1) d\sigma(x_2)$$

depends meromorphically on α_3 . At $\alpha_3 = -\rho - 2k$, the expression has a simple pole and its residue is given by

$$\int_S R_k[f_1(\cdot) f_2(x) F_3(\cdot, x)](x) d\sigma(x)$$

Now, we differentiate under the integral sign (assuming, as it is possible, that F_3 is $2k$ -times continuously differentiable), to get

$$\begin{aligned} & \text{Res}(K_\alpha(f_1, f_2, f_3), \alpha^0) \\ = & \\ & = \int_{S \times S} f_2(x) f_3(x_3) R_k[f_1(\cdot) |x_3 - \cdot|^{-\rho+\alpha_2}](x) |x - x_3|^{-\rho+\alpha_1} d\sigma(x_3) d\sigma(x) . \end{aligned}$$

The equality (24) is thus proved for $\Re\alpha_1$ and $\Re\alpha_2$ large enough. \square

The left handside of (24) is *a priori* defined in the intersection of the domain of definition of K_α and the plane $\alpha_3 = -\rho - 2k$, that is to say on \mathbb{C}^2 only outside of the lines

$$\alpha_1 = -\rho - 2l_1, \quad \alpha_2 = -\rho - 2l_2, \quad \alpha_1 + \alpha_2 = 2k - 2l_3,$$

where l_1, l_2, l_3 are in \mathbb{N} . A consequence of Theorem 5.1 is that it can be extended to a larger domain. This can be illustrated on the evaluation of K_α for f_1, f_2, f_3 equal to the constant function 1. Then (see [2]), up to a constant, $K_\alpha(1, 1, 1)$ is equal to

$$\frac{\Gamma\left(\frac{\alpha_1+\alpha_2+\alpha_3+\rho}{2}\right) \Gamma\left(\frac{\alpha_1+\rho}{2}\right) \Gamma\left(\frac{\alpha_2+\rho}{2}\right) \Gamma\left(\frac{\alpha_3+\rho}{2}\right)}{\Gamma\left(\rho + \frac{\alpha_2+\alpha_3}{2}\right) \Gamma\left(\rho + \frac{\alpha_3+\alpha_1}{2}\right) \Gamma\left(\rho + \frac{\alpha_1+\alpha_2}{2}\right)} .$$

The residue at $\alpha_3 = -\rho - 2k$ is equal to

$$\frac{(-1)^k}{k!} \frac{\Gamma\left(\frac{\alpha_1+\alpha_2}{2} - k\right) \Gamma\left(\frac{\alpha_1+\rho}{2}\right) \Gamma\left(\frac{\alpha_2+\rho}{2}\right)}{\Gamma\left(\frac{\rho+\alpha_2}{2} - k\right) \Gamma\left(\frac{\rho+\alpha_1}{2} - k\right) \Gamma\left(\rho + \frac{\alpha_1+\alpha_2}{2}\right)} .$$

which equals

$$\frac{(-1)^k}{k!} \left(\frac{\rho + \alpha_1}{2} - 1\right) \dots \left(\frac{\rho + \alpha_1}{2} - k\right) \left(\frac{\rho + \alpha_2}{2} - 1\right) \dots \left(\frac{\rho + \alpha_2}{2} - k\right) \frac{\Gamma\left(\frac{\alpha_1+\alpha_2}{2} - k\right)}{\Gamma\left(\rho + \frac{\alpha_1+\alpha_2}{2}\right)}$$

and this expression has simple poles exactly along the lines

$$\alpha_1 + \alpha_2 = 2k - 2l, l \in \mathbb{Z} .$$

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